ZERO AND INFINITY.

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An objection has been raised against the conclusions reached in my article on Zero and Infinity, in the July No. of the ANALYST, that if these interpretations be admitted, then we must reject many very beautiful and interesting generalizations of Analytical Geometry. Thus,— "Every strai't line meets a curve of the second degree in two distinct, coincident, or imaginary points." (See Salmon's Conic Sections, 5th edition, Art. 135.)

Let us take the equations

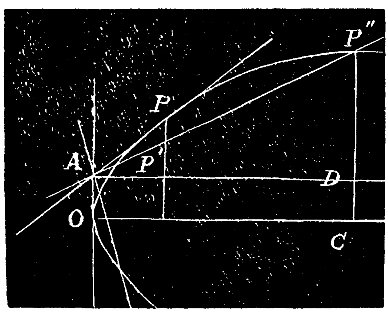
$$y = nx + a, \tag{1}$$

$$y^2 = 4ax. \tag{2}$$

Eliminating x we find for the ordinate of the point of intersection

$$y = \frac{2a}{n}\left(1 \pm \surd[1-n]\right). \tag{3}$$

If $n > 1$, the line meets the curve in two imaginary points: if $n = 1$, in two coincident points at P, and is a tangent: if $n < 1$, in two distinct p'ts at P' and P'': if $n = 0$ the line becomes parallel to the axis, and $y = 2a.\frac{0}{0}$, which by (1) is $y = a$; or $y = 4a \div 0$, and the second point of intersection is said to be "at infinity". But how can there be a real point of intersection at infinity when DC remains constant and equal to a, while CP'' increases without limit? Can we believe that the two p'ts D and P'', though infinitely distant from each other, are yet coincident?



It would seem that the only intelligible explanation is, that if n is an infinitesimal, the line AP'' is not quite paral'l to OC; and as the ordinate to this right line increases proportionally to its abscissa, while that to the curve increases proportionally to the square root of its abscissa, therefore the ordinate to the right line will eventually become the greater, and there will be a point of intersection indefinitely remote from the origin; but if $n = 0$ then $4a \div 0$ is a symbol of impossibility and there can be no second point of intersection.

If n is negative there will be two real points of intersection, whose abscissas are, one positive and one negative, each approaching zero as n increases. This indicates two coincident points at O when $n = -\infty$.

Again, let us consider an hyperbola

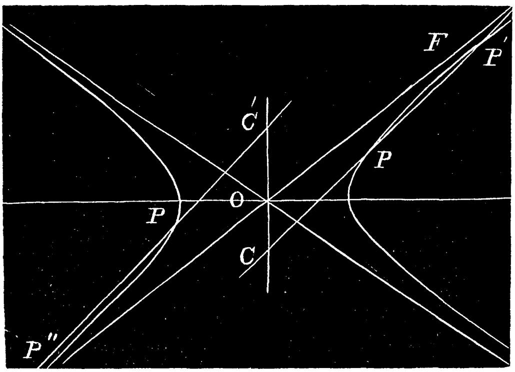
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

and a right line

$$y = nx + c.$$

If c is negative and $n < (b \div a)$ we have two distinct points of intersection at P and P'.

If n is very nearly equal to $b \div a$ P' will be indefinitely remote from the origin. If now $c, = OC$, decreases indefinitely the line PP' will approach indefinitely to coincidence with the asymptote OT, and PP' may be said to meet the curve in two coincident points at positive infinity. If c is positive the point of intersection will be on the left hand branch of the curve at minus infinity.



According to the older view, a line through the origin ($y = nx$) meets the curve in two imaginary points if $n > (b \div a)$. If $n < (b \div a)$ it meets the curve in two distinct points, one in the positive direction as at P', the other in the negative, as P''. Now when $n = (b \div a)$ the coordinates of P' become $+$ infinity while those of P'' become — infinity, and since $a \div (+0) = a \div (-0)$, plus infinity and minus infinity have been supposed to be two coincident points. This is manifestly absurd. The true statement is: (1), An asymptote to an hyperbola does not meet the curve. (2), Any other line through the centre meets the curve in two imaginary, or in two distinct points (the coordinates of one being $+$, those of the other $-$). (3), A line passing very near the centre and very nearly parallel to the asymptote meets the curve in two p'ts very remote and very nearly coincident.

Once more—"A right line meets a curve of the third degree in three p'ts, one of which must be real; and an asymptote to a curve of the third degree must meet the curve in one real point besides the two coincident points at infinity". (See Salmon's Higher Plane Curves, Chapt. III; also, Williamson's Diff. Calc., Art. 197.)

Let us take the curve $\quad x^3 + y^3 = a^3.$

The equation of its asymptote is

$$y + x = 0.$$

Now what are the three *real* points of intersection of the asymptote with the curve? If we substitute the value of $y = -x$ in the first equation, we have

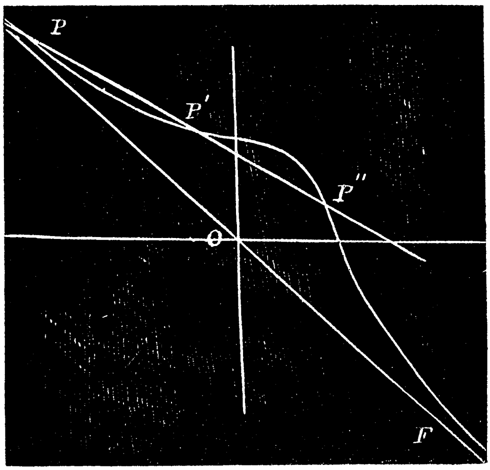
$$x^3 - x^3 = a^3, \text{ or } 0x^3 = a^3, \text{ or } x^3 = \frac{a^3}{0}.$$

What are the three roots of this equation? It may be written

$$0x^3 + 0x^2 + 0x = a^3,$$

which is said th have *"three roots, each equal to infinity"*, thus giving three points of intersection. This is to the writer *profoundly obscure.*

The line $y = nx + c$, if $c < a$, meets the curve in 3 distinct p'ts, P', P'', P'''.



If this line is very nearly parallel to the asymptote P will be very remote from the origin.

If now c decreases indefinitely, P' will approach P while P'' will recede indefinitely toward T, and the asymptote is the limiting position of $P\ P'\ P''$. Hence the line $y = nx + c$ may be said to meet the curve in two coincident points infinitely remote on one side of the origin, and at a distinct point, also infinitely remote, on the other side, if c is an infinitesimal ($c = \bigcirc$) and $n = \pm \bigcirc$.

Hence we adhere to the interpretations

$$\frac{a}{0} = \text{impossibility; and } \frac{a}{\bigcirc} = \infty.$$

NOTE.—The value of Ψ, at p. 172, should be

$$\Psi = A\frac{d\Phi^2}{dx^2} + B\frac{d\Phi^2}{dy^2} + C\frac{d\Phi^2}{dz^2} + \text{\&c.}$$

The error was made by me, in copying

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